MANAV RACHNA Vidyanatariksha | MANAV RACHNA UNIVERSITY (FORMERLY MANAV RACHNA COLLEGE OF ENGINEERING NAAC ACCREDITED 'A' GRADE INSTITUTION)
Declared as State Private University under section 2f of the UGC act, 1956

DEPARTMENT OF MATHEMATICS

"T3-Examination, May-2018"

Semester:2nd
Subject:Measure Theory
Branch: Mathematics
Course Type: Core
Time: 180 Min
Max.Marks: 100

Date of Exam:15/05/2018
Subject Code:MAH512
Session: I
Course Nature: Soft
Program: M.Sc.
Signature: HOD/Associate HOD:

Note: Attempt Any Two Questions From Each Part

Part-A

1. (a) Let $(X, \mathcal{S})$ be a measurable space and $A \in \mathcal{S}$. Then prove that for any set B; $\mu^*(A \cup B) + \mu^*(A \cap B) = \mu^*(A) + \mu^*(B)$. (4)
(b) Define a measurable space. Let μ be a measure on an algebra $\mathcal{A}$ of subsets of set X and let μ^* be the induced outer measure. Let $E \in \mathcal{S}^*$ be such that $\mu^*(E) < +\infty$. Then prove that for given $\epsilon > 0, \exists$ a set $F_\epsilon \in \mathcal{A}$ such that $\mu^*(E \Delta F_\epsilon) < \epsilon$. (6)

2. (a) Define Lebesgue measure and Borel Sets. (2)
(b) Prove that Cantor's ternary set is uncountable, but it's measure is zero. (8)

3. (a) Define Measurable Cover and Measurable Kernel. (2)
(b) The map $E \mapsto E + x$ be a homeomorphic map from $\mathbb{R} \to \mathbb{R}, \forall\, E \subseteq \mathbb{R}$. Then prove that $\lambda: \mathbb{R} \to \mathbb{R}$ is a unique translational invariant measure on $\mathbb{R}$ with $\lambda([0,1]) = 1$. (8)

Part-B

4. (a) Sate and prove Lebesgue Dominated Convergence theorem. (8)
(b) If $\{s_n\}_{n\geq 1}$ be an increasing sequence of non-negative simple measurable function in measure space X such that $\lim_{n\to\infty} s_n(x) = s(x), \forall\, x \in X$. Then prove that $\int s\, d\mu = \lim_{n\to\infty} \int s_n d\mu$. (5)
(c) Prove that $\int s\, d\mu = \sup\{\int s' d\mu; 0 \leq s' \leq s,\ s', s \in \mathcal{L}_0^+\}$. (3)
(d) Define Lebesgue integral of non-negative measurable function and prove that $\mathcal{L}_0^+ \subseteq \mathcal{L}^+$. (4)

5. (a) Let $(X, \mathcal{S}, \mu)$ be a measure space and $f: X \to \mathbb{R}^*$ be a $\mathcal{S}$ −measurable function. Then $\forall\, E \in \mathcal{S}$, prove that $\chi_E f \in \mathcal{L}^+$. If $\nu(E) \coloneqq \int_E f\, d\mu$, then prove that ν is a measure on $\mathcal{S}$ and $\nu(E) = 0$, whenever $\mu(E) = 0$. (10)
(b) Let $f: [a, b] \to \mathbb{R}$ be a Riemann integrable function. Then prove that $f \in \mathcal{L}_1[a, b]$, i.e. f is Lebesgue integrable and $\int_{[a,b]} f\, d\lambda = \int_a^b f\, dx$. (10)

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6. (a) Let $(X, \mathcal{S}, \mu)$ be a measure space and $\mathcal{L}_1$ be a set of all μ −integrable function. Let $f \in \mathcal{L}_1$ and $E_i \in \mathcal{S}, i \geq 1$ such that $E_i \cap_{i \neq j} E_j = \phi$. Then prove that $\sum_{i=1}^{\infty} \int_{E_i} f \, d\mu$ is absolutely convergent and if $E = \cup_{i=1}^{\infty} E_i$, then $\sum_{i=1}^{\infty} \int_{E_i} f \, d\mu = \int_E f d\mu$. (10)

(b) State and prove Monoton Convergence Theorem. (10)

Part-C

7. (a) Define absolute continuity of a function. Prove that every absolute continuous function is a function of bounded variation. (6)

(b) Let $(X, \mathcal{S}, \mu)$ be a measure space and $\mathcal{L}_1$ be a set of all μ −integrable function. If $f \in \mathcal{L}_1[a, b]$ and $\int f \, d\mu = 0$, then prove that $f(x) = 0 \; a.e.$ (4)

(c) Let $f: [a, b] \to \mathbb{R}$ be a monotonic increasing function. Then prove that $f' \in \mathcal{L}_1[a, b]$ and $\int_{[a,b]} f \, d\lambda \leq f(b) - f(a)$. (10)

8. (a) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces, $f \in \mathcal{L}_1(\mu)$ and $g \in \mathcal{L}_1(\nu)$. Let $\phi(x, y) = f(x). g(x), \; \forall \, x \in X, y \in Y$. Then prove that (10)

   (i) $\phi \in \mathcal{L}_1(\mu \times \nu)$

   (ii) $\int_{X \times Y} \phi(x, y) \, d(\mu \times \nu) = \left(\int_X f(x) d\mu\right). \left(\int_Y g(y) d\nu\right)$.

(b) Define product measure. Let $p_X: X \times Y \to X$ and $p_Y: X \times Y \to Y$ be defined by $p_X(x, y) = x$ and $p_Y(x, y) = y, \forall \, x \in X, y \in Y$. Then prove the following holds

(i) p_X and p_Y is measurable.

(ii) The σ −algebra $\mathcal{A} \otimes \mathcal{B}$ is the smallest σ −algebra generated by the subsets of $X \times Y$ such that (i) holds. (10)

9. (a) Define absolute continuous measure. Let $f: \mathbb{R} \to \mathbb{R}$ be a monotonically increasing right continuous function and μ_f be the measure induced by f on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Then prove that $\mu_f \ll \lambda$ if and only if f is absolutely continuous on every bounded interval. (10)

(b) State and prove Lebesgue Decomposition Theorem. (10)

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