MANAV RACHNA vidyanatariksha | MANAV RACHNA UNIVERSITY
(FORMERLY MANAV RACHNA COLLEGE OF ENGINEERING
NAAC ACCREDITED 'A' GRADE INSTITUTION)
Declared as State Private University under section 2f of the UGC act, 1956

DEPARTMENT OF MATHEMATICS.

"T3, May 2018"

Semester: II | **Date of Exam**: 19/05/2018
Subject: Advanced Linear Algebra | **Subject Code**: MAH 510
Branch: M.Sc. Mathematics | **Session**: Morning
Course Type: Core | **Course Nature: Hard**
Time: 3 Hours | **Program: M.SC**
Max.Marks: 100 | **Signature: HOD/Associate HOD:**

Note: Part A: Attempt any two. Part B: Attempt any two. Part C: attempt any two.

Part – A

Q1 Define minimal polynomial of a linear operator. Prove that the minimal polynomial of a matrix is a divisor of every polynomial that annihilates the matrix. (10)

Q2 State and prove Sylvester's law of nullity. (10)

Q3 a) If T is any linear operator on a vector space V, then prove that the range of T and the null space of T are both invariant under T. (5)

b) Show that the space generated by (1, 1, 1) and (1, 2, 1) is an invariant subspace of R^3 under T, where T(x, y, z) = (x + y – z, x + y, x + y – z). (5)

Part – B

Q4a) State and prove Cauchy – Schwarz Inequality. (10)

b) Prove that for any linear operator T on a finite dimensional inner product space V ∋ a unique linear operator T^* on V such that $\langle Tu, v\rangle = \langle u, T^*v\rangle$ for all u, v in V. (10)

Q5a) Let T be a linear operator on an inner product space V. Then T is unitary if and only if the adjoint T^* of T exists and $T T^* = T^*T = I$. (5)

b) Prove that to every normal operator T on a finite dimensional complex inner product space V there correspond distinct complex numbers $c_1, c_2, \ldots, c_k$ and perpendicular projections $E_1, E_2, \ldots, E_k$ so that

(i) the E_i are pairwise orthogonal and different from zero.

(ii) $E_1 + E_2 + \ldots + E_k = I$

(iii) $T = c_1E_1 + c_2E_2 + \ldots + c_kE_k$. (15)

Q6 a) Let V be an inner product space and W be subspace of V. Then prove that $V = W \oplus W^{\perp}$. (6)

b) If in an inner product space $||u + v|| = ||u|| + ||v||$, then prove that the vectors u and v are linearly dependent. (4)

c) Prove that any orthonormal set of vectors in an inner product space is linearly dependent. (5)

d) Apply the Gram Schmidt process to the vectors (1, 0, 2, 0) , (1, 2, 3, 1), to obtain an orthonormal basis for R^4 with the standard inner product. (5)

Part – C

Q7a) If U is an n – dimensional vector space with basis $\{u_1, u_2,\dots u_n\}$, if V is an m – dimensional vector space with basis $\{v_1, v_2,\dots v_m\}$, and if $\{a_{ij}\}$ is any set of nm scalars (i = 1, 2,…n; j = 1,…m) then there is one and only one bilinear form on $U \oplus V$ such that $f(u_i, v_j) = a_{ij}$ for all i and j. (8)

b) Let F be the bilinear form on R^2 defined by $f((x_1,x_2), (y_1, y_2)) = (x_1+x_2)(y_1+y_2)$. Find the matrix of f in the ordered basis B = {(1, -1), (1, 1)}. (3)

c) Is the bilinear form $2x_1y - x_1y_2 + x_2y_2 - x_3y_3$ symmetric ? Validate your answer. (2)

d) Reduce the form $X'AY$ into canonical form where $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$. Also find the equations of transformations. (7)

Q8 a) Prove that every bilinear form on the vector space V over a subfield F of the complex numbers can be uniquely expressed as the sum of a symmetric and skew symmetric bilinear forms. (8)

b) Diagonalize the quadratic form

$$x_1^2 - 4x_2^2 + 6x_3^2 + 2x_1x_2 - 4x_1x_3 + 2x_4^2 - 6x_3x_4.$$

Find the equations of linear transformations. Also find the rank, index and signature of the form. (12)

Q9 a) Let f be a non – degenerate bilinear form on a finite dimensional vector space V. Prove that the set G of all linear operators on V which preserve f is a group under the operation of composition. (8)

b) Show that the quadratic form $-21x_1^2 + 30x_1x_2 - 12x_1x_3 - 11x_2^2 + 8x_2x_3 - 2x_3^2$ is negative semi – definite using diagonalisation process. (12)